

# Three loop $\overline{\text{MS}}$ renormalization of QED in the 't Hooft-Veltman gauge

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**Abstract.** Quantum electrodynamics (QED) fixed in the 't Hooft-Veltman gauge is renormalized to three loops in the  $\overline{\text{MS}}$  scheme. The  $\beta$ -functions and anomalous dimensions are computed as functions of the usual QED coupling and the additional coupling,  $\xi$ , which is introduced as part of the nonlinear gauge fixing condition. Similar to the maximal abelian gauge of quantum chromodynamics, the renormalization of the gauge parameter is singular.

# 1 Introduction.

Gauge theories are the underlying quantum field theory describing the physics of the fundamental properties of nature. For instance, the field theory describing the strong interaction, quantum chromodynamics (QCD), is based on a non-abelian colour group and exhibits asymptotic freedom, [1, 2], whereby the basic quarks and gluons effectively behave as free particles at very high energies. Indeed this property allows one to apply perturbation theory to describe strong physics phenomena such as deep inelastic scattering to very high precision. One fundamental reason for the basic feature of asymptotic freedom, [1, 2], is the self-interaction of the gluon which is a natural consequence of the generalization of the abelian gauge theory of quantum electrodynamics (QED) to the non-abelian gauge group. The consequent nonlinearities introduced in this extension provide a richer structure not present in the dynamics of electrons and photons. In order to perform quantum calculations to determine the properties of a gauge theory, one has to first fix the gauge to ensure that only the physical degrees of freedom are taken into account. There is a large range of choices as to how to eliminate the unphysical degrees of freedom. In general such choices can be classified roughly under various headings, such as covariant or non-covariant, linear or nonlinear and physical or unphysical. (For a comprehensive review see, for example, [3].) Ordinarily when one wishes to perform high precision and therefore high loop calculations, one chooses linear covariant gauges (which are not physical) such as the Landau or Feynman gauge. This is primarily because their covariant, though unphysical, nature does not overcomplicate the resultant (massless) Feynman diagrams. By contrast other gauges, whilst being motivated by physical considerations, such as the Coulomb gauge, are not necessarily renormalizable. If they can be proved to be renormalizable at a formal level, then it is not always clear whether loop integrals beyond one loop can be computed, [3]. However, irrespective of how one fixes the gauge, one fundamental feature is always present and that is that physical predictions must always be independent of the choice of gauge. As the bulk of multiloop calculations are performed in linear covariant gauges and mass independent renormalization schemes, one check in this instance is that the covariant gauge fixing parameter,  $\alpha$ , must be absent in the determination of physical quantities or objects which are clearly gauge independent or gauge invariant, [4]. Indeed this property provides a powerful checking tool in intricate multiloop calculations.

Although linear covariant gauges have been examined in detail, there has been recent interest in *nonlinear* covariant gauges due to their potential connection with the infrared dynamics of non-abelian gauge theories, [5, 6]. For instance, the Curci-Ferrari gauge, [7], and maximal abelian gauge (MAG), [8, 9, 10], have been studied where the aim was to examine the abelian dominance hypothesis, [8, 11, 12, 13]. Briefly the gluons in the centre of the colour group are believed to dominate the infrared sector due to the off-diagonal gluons acquiring a dynamical mass greater than that of the centre gluons. The latter are then central to abelian monopole condensation that is believed to drive the confinement mechanism, [8, 11, 12, 13]. However, from a practical point of view to perform any calculations in such nonlinear gauges one needs to renormalize the gauge theory to as high a loop order as is possible. This has been achieved for both the Curci-Ferrari gauge and MAG in the  $\overline{\text{MS}}$  scheme at three loops, [14, 15]. From the point of view of trying to understand basic features of covariant nonlinearly gauge fixed gauge theories both these gauges have common properties. For instance, unlike linear covariant gauges they have quartic ghost interactions and the corresponding gauge parameters get non-trivial renormalization. Further in the case of the MAG this gauge parameter renormalization is singular. Though gauge independent quantities, such as the  $\beta$ -function, correctly emerge as gauge parameter independent. Whilst these gauges are essentially related by construction to the non-abelian aspect of the gauge theory itself, one natural question to ask is, is this a feature

in other nonlinear gauges when one has an abelian structure.

Clearly QED is invariably treated in a linear covariant gauge. However, with the explosion of interest in gauge theories in the early 1970's QED was gauge fixed in a nonlinear gauge known as the 't Hooft-Veltman gauge, [16], and shown to be renormalizable. The primary interest in this gauge fixing was that the nonlinearity naturally introduced an abelian gauge theory which mimicked QCD. This was due not only to the presence of interacting Faddeev-Popov ghosts but also triple and quartic *photon* self-interactions. As the latter do not appear in linear covariant gauges in QED, the 't Hooft-Veltman gauge clearly could be used as a laboratory to study simple issues related to gauge field self-interaction. Indeed in [17, 18] a one loop calculation showed that the corresponding covariant gauge fixing parameter was renormalized. Therefore, in light of these observations it is the purpose of this article to record the full three loop renormalization of QED in the 't Hooft-Veltman gauge. Indeed as far as we are aware this will represent the *first* detailed multiloop study of the renormalization of QED in the 't Hooft-Veltman gauge. We will construct all the renormalization group functions in the  $\overline{\text{MS}}$  scheme including the  $\beta$ -function of the electron-photon coupling constant which will agree with the already established results of [19, 20, 21, 22].

The paper is organised as follows. The relevant properties of QED gauge fixed in the 't Hooft-Veltman gauge are discussed in section two with the details of the full renormalization given in section three. Concluding comments are provided in section four.

## 2 Background.

First, we introduce the 't Hooft-Veltman gauge in QED, [16], and the notation and conventions we will use. The key ingredient is the gauge fixing functional,  $\mathcal{F}[A_\mu]$ , which slots into the conventional path integral formalism for constructing a quantized gauge theory. Here  $A_\mu$  is the photon field. We take

$$\mathcal{F}[A_\mu] = \partial^\mu A_\mu + \frac{1}{2}\xi A^\mu A_\mu \quad (2.1)$$

which is clearly nonlinear where for the moment  $\xi$  is a parameter and  $\alpha$  is the gauge fixing parameter. Clearly when  $\xi = 0$  one recovers the usual linear gauge fixing functional whence  $\alpha$  becomes equivalent to the gauge parameter of those gauges. In other studies of the 't Hooft-Veltman gauge, however,  $\xi$  was invariably fixed to certain numerical values such as 1 or 2. We leave it as a free parameter here and given that eventually it will appear with the triple and quartic photon self-interactions we will regard it as a coupling constant which will run. It is not to be confused with the usual gauge coupling constant,  $e$ , which is present in the covariant derivative when electrons are present. Therefore we are in effect working with a two coupling theory. Though in the absence of electrons, whilst photon self-interactions are present the field theory is effectively a free theory of photons since the physics cannot be altered by the gauge fixing. This feature ought to emerge in the computations. Hence the full Lagrangian for  $N_f$  massless electrons in the 't Hooft-Veltman gauge is, [16],

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{c}\partial^\mu\partial_\mu c + \xi\bar{c}A^\mu\partial_\mu c + b\left(\partial^\mu A_\mu + \frac{1}{2}\xi A^\mu A_\mu\right) + \frac{1}{2}\alpha b^2 + i\bar{\psi}\not{D}\psi \quad (2.2)$$

where  $\psi$  is the electron field,  $c$  and  $\bar{c}$  are the Faddeev-Popov ghosts emerging from the path integral formalism and  $b$  is the Nakanishi-Lautrup auxiliary field which arises in the off-shell BRST formalism. Eliminating it by its equation of motion produces the Lagrangian in the form we will treat it

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{c}\partial^\mu\partial_\mu c + \xi\bar{c}A^\mu\partial_\mu c - \frac{1}{2\alpha}\left(\partial^\mu A_\mu + \frac{1}{2}\xi A^\mu A_\mu\right)^2 + i\bar{\psi}\not{D}\psi. \quad (2.3)$$

Ordinarily in a linear covariant gauge in QED one drops the Faddeev-Popov ghosts from (2.3) when  $\xi = 0$  since they do not couple to photons or electrons. For  $\xi \neq 0$  this is not possible and they are not only present but play a key role in the full renormalization of the theory. The covariant derivative,  $D_\mu$ , is defined by

$$D_\mu = \partial_\mu + ieA_\mu . \quad (2.4)$$

Unlike the Curci-Ferrari gauge and the MAG in QCD, there is no quartic ghost self-interaction which is due to the absence of a colour index on the ghost fields meaning that  $c(x)c(x) = 0$  due to their anticommuting property. The Feynman rules for (2.3) are straightforward to derive but the interested reader can view them in [23]. By construction (2.2) is invariant under the BRST symmetry, [18],

$$\delta A_\mu = -\partial_\mu c , \quad \delta c = 0 , \quad \delta \bar{c} = b , \quad \delta b = 0 , \quad \delta \psi = iec\psi \quad (2.5)$$

which is clearly nilpotent.

### 3 Renormalization.

We turn now to the details of our three loop  $\overline{\text{MS}}$  renormalization. If we regard the fields and parameters of (2.3) as bare then to renormalize (2.3) we introduce renormalized fields and variables by

$$\begin{aligned} A_o^\mu &= \sqrt{Z_A} A^\mu , \quad c_o = \sqrt{Z_c} c , \quad \bar{c}_o = \sqrt{Z_{\bar{c}}} \bar{c} , \quad \psi_o = \sqrt{Z_\psi} \psi , \\ e_o &= \mu^\epsilon Z_e e , \quad \xi_o = \mu^\epsilon Z_\xi \xi , \quad \alpha_o = Z_\alpha^{-1} Z_A \alpha \end{aligned} \quad (3.1)$$

where the subscript  $_o$  denotes a bare quantity. We have also chosen to follow the same convention as [14] for the definition of the renormalization of the gauge parameter  $\alpha$ . With this convention it is the combination  $Z_A Z_\alpha^{-1}$  which is unity in the linear covariant gauge after renormalization. As we will use dimensional regularization in  $d = 4 - 2\epsilon$  dimensions with  $\epsilon$  as the regularizing parameter, the renormalization scale  $\mu$  has been introduced to ensure that both renormalized couplings  $e$  and  $\xi$  remain dimensionless in  $d$ -dimensions. In principle all the renormalization constants will be functions of both (renormalized) couplings. However, it transpires that both the  $\beta$ -function of  $e$  and the photon anomalous dimension are independent of  $\xi$ . Therefore, the information determined from the  $\overline{\text{MS}}$  renormalization will be encoded in the renormalization group functions. The  $\beta$ -functions are given by

$$\begin{aligned} \mu \frac{\partial a}{\partial \mu} &= \beta_a(a) = \frac{1}{2}(d-4)a - 2a\beta_a(a) \frac{\partial}{\partial a} \ln Z_e \\ \mu \frac{\partial z}{\partial \mu} &= \beta_z(a, z) = \frac{1}{2}(d-4)z - 2z\beta_a(a) \frac{\partial}{\partial a} \ln Z_\xi - 2z\beta_z(a, z) \frac{\partial}{\partial z} \ln Z_\xi . \end{aligned} \quad (3.2)$$

The anomalous dimensions are defined in the usual way by, (see, for example, [24]),

$$\gamma_A(a) = \frac{\partial \ln Z_A}{\partial \ln \mu} , \quad \gamma_\alpha(a, z) = \frac{\partial \ln \alpha}{\partial \ln \mu} , \quad \gamma_c(a, z) = \frac{\partial \ln Z_c}{\partial \ln \mu} , \quad \gamma_\psi(a, z) = \frac{\partial \ln Z_\psi}{\partial \ln \mu} \quad (3.3)$$

from which it is straightforward to deduce

$$\begin{aligned} \gamma_A(a) &= \beta_a(a) \frac{\partial}{\partial a} \ln Z_A + \beta_z(a, z) \frac{\partial}{\partial z} \ln Z_A + \alpha \gamma_\alpha(a, z) \frac{\partial}{\partial \alpha} \ln Z_A \\ \gamma_\alpha(a, z) &= \left[ \beta_a(a) \frac{\partial}{\partial a} \ln Z_\alpha + \beta_z(a) \frac{\partial}{\partial z} \ln Z_\alpha - \gamma_A(a) \right] \left[ 1 - \alpha \frac{\partial}{\partial \alpha} \ln Z_\alpha \right]^{-1} \end{aligned}$$

$$\begin{aligned}
\gamma_c(a, z) &= \beta_a(a) \frac{\partial}{\partial a} \ln Z_c + \beta_z(a, z) \frac{\partial}{\partial z} \ln Z_c + \alpha \gamma_\alpha(a, z) \frac{\partial}{\partial \alpha} \ln Z_c \\
\gamma_\psi(a, z) &= \beta_a(a) \frac{\partial}{\partial a} \ln Z_\psi + \beta_z(a, z) \frac{\partial}{\partial z} \ln Z_\psi + \alpha \gamma_\alpha(a, z) \frac{\partial}{\partial \alpha} \ln Z_\psi
\end{aligned} \tag{3.4}$$

in terms of the renormalization constants where we have set  $a = e^2/(16\pi^2)$  and  $z = \xi^2/(16\pi^2)$ . In these definitions, (3.4), we have not assumed that  $\gamma_\alpha(a, z) = 1$  which is the case in a linear covariant gauge. Clearly the wave function renormalizations will also depend on  $\alpha$ . Moreover in these definitions where our explicit renormalization constants are clearly independent of one of the coupling constants or gauge parameter, we have included this property in the derivation of (3.2) and (3.4). For instance,  $Z_e$  turns out to be independent of both  $a$  and  $\alpha$ .

Green's function	One loop	Two loop	Three loop	Total
$A_\mu A_\nu$	3	18	254	275
$c \bar{c}$	1	6	78	85
$\psi \bar{\psi}$	1	6	78	85
$A_\mu \bar{c} c$	2	33	688	723
$A_\mu \bar{\psi} \psi$	2	33	688	723
Total	9	96	1786	1891

Table 1. Number of Feynman diagrams for the renormalization of each Green's function.

The full three loop renormalization is performed for the massless case using the MINCER algorithm, [24, 25], written in the symbolic manipulation language FORM, [26]. The Feynman diagrams are generated automatically with the QGRAF package, [27], before being converted into FORM input notation by converter routines. The number of Feynman diagrams we compute at each loop order for the set of Green's functions we need to consider to render (2.3) finite are given in Table 1. The wave function renormalizations are deduced from the photon, Faddeev-Popov and electron 2-point functions whilst the 3-point functions determine the coupling constant renormalizations. For the latter given that the coupling constant renormalizations are gauge independent we have a strong check on the wave function calculations. Also for these, to apply the MINCER algorithm an external momentum has to be nullified. This is because MINCER computes massless 2-point functions up to the finite part at three loops. The extraction of each of the renormalization constants is found by applying the approach of [28] for automatic Feynman diagram calculations. The 2 or 3-point functions are computed as a function of the bare parameters. Then these are replaced by the renormalized variables from (3.1) and the undetermined renormalization constant for that 2 or 3-point function chosen so as to absorb the infinities which remain. The latter appear as poles in  $\epsilon$  and are absorbed into the renormalization constants with the usual  $\overline{\text{MS}}$  definition. Prior to presenting the results of our labours we note that one main check is that the double pole in  $\epsilon$  at two loops and the double and triple poles at three loops for any renormalization constant are predetermined by the structure of the renormalization group equation. In the expressions we present for the anomalous dimensions and  $\beta$ -functions all the renormalization constants passed this test.

Hence, the complete set of three loop  $\overline{\text{MS}}$  renormalization group functions are

$$\begin{aligned}
\beta_a(a) &= \frac{1}{2}[d-4]a + \frac{4}{3}N_f a^2 + 4N_f a^3 - \left[ \frac{44}{9}N_f^2 + 2N_f \right] a^4 + O(a^5) \\
\beta_z(a, z) &= \frac{1}{2}[d-4]z + \frac{4}{3}N_f z a + 4N_f z a^2 - \left[ \frac{44}{9}N_f^2 + 2N_f \right] z a^3 + O(z a^4) \\
\gamma_A(a) &= \frac{4}{3}N_f a + 4N_f a^2 - \left[ \frac{44}{9}N_f^2 + 2N_f \right] a^3 + O(a^4)
\end{aligned}$$

$$\begin{aligned}
\gamma_\alpha(a, z) &= -\frac{4}{3}N_f a - [2\alpha^2 - 3\alpha + 3]\frac{z}{2\alpha} - 4N_f a^2 - [5\alpha - 16]\frac{N_f z a}{3\alpha} \\
&\quad - [21\alpha^3 - 20\alpha^2 - 29\alpha + 60]\frac{z^2}{16\alpha} + \left[\frac{44}{9}N_f^2 + 2N_f\right]a^3 \\
&\quad - [(140\alpha - 240)N_f^2 + (1215\alpha - 1296\zeta(3)\alpha - 6210 + 5184\zeta(3))N_f]\frac{za^2}{54\alpha} \\
&\quad - [30\alpha^2 - 61\alpha - 31]\frac{N_f z^2 a}{8\alpha} \\
&\quad - [264\zeta(3)\alpha^4 + 370\alpha^4 - 48\zeta(3)\alpha^3 - 116\alpha^3 - 1008\zeta(3)\alpha^2 \\
&\quad \quad - 1093\alpha^2 + 432\zeta(3)\alpha + 1778\alpha + 360\zeta(3) - 367]\frac{z^3}{128\alpha} + O(z^n a^{4-n}) \\
\gamma_c(a, z) &= \frac{1}{4}[3 - \alpha]z - \frac{5}{6}N_f z a - \frac{1}{32}[5\alpha^2 - 16\alpha - 5]z^2 \\
&\quad - [140N_f^2 + (1215 - 1296\zeta(3))N_f]\frac{za^2}{108} + [13 - 17\alpha]\frac{N_f z^2 a}{16} \\
&\quad - [111\alpha^3 - 48\zeta(3)\alpha^2 - 28\alpha^2 - 192\zeta(3)\alpha - 273\alpha - 144\zeta(3) + 410]\frac{z^3}{256} \\
&\quad + O(z^n a^{4-n}) \\
\gamma_\psi(a, z) &= \alpha a - [4N_f + 3]\frac{a^2}{2} - [3\alpha^2 - 4\alpha + 1]\frac{za}{4} + [40N_f^2 + 54N_f + 27]\frac{a^3}{18} \\
&\quad + [(9\alpha - 20)N_f - 8\alpha^3 + 24\zeta(3)\alpha - 32\alpha - 72\zeta(3) + 72]\frac{za^2}{4} \\
&\quad - [336\zeta(3)\alpha^3 - 323\alpha^3 - 432\zeta(3)\alpha^2 + 480\alpha^2 + 48\zeta(3)\alpha \\
&\quad \quad - 197\alpha + 432\zeta(3) - 344]\frac{z^2 a}{64} + O(z^n a^{4-n}) . \tag{3.5}
\end{aligned}$$

where the formal order symbol  $O(z^n a^{l-n})$  means all appropriate possible combinations of the coupling constants  $a$  and  $z$  at the  $l$ th loop. In both  $\beta$ -functions the  $d$ -dimensional dependence has been retained as an indication of our conventions in deriving the renormalization group functions as well as for the reader interested in constructing the original renormalization constants from the differential equations of (3.2) and (3.4).

Aside from the internal checks based on consistency with the renormalization group equation there are additional checks on these results. First, both  $\beta$ -functions correctly emerge as  $\alpha$  independent and  $\beta_a(a)$  is in total agreement with the original linear covariant gauge result of [19, 20, 21, 22]. Also the photon anomalous dimension is proportional to  $\beta_a(a)$  as required by the Ward identity. Further, in the limit  $z \rightarrow 0$  the anomalous dimensions agree with those of the linear covariant gauge fixing, [28]. Finally, one must recover a theory of free photons when the electron interaction is switched off via  $a \rightarrow 0$ . Clearly,  $\beta_z(0, z) = 0$  which corresponds to a free field theory when  $\xi \neq 0$  even though neither  $\gamma_\alpha(0, z)$ ,  $\gamma_c(0, z)$  nor  $\gamma_\psi(0, z)$  are zero. An additional comment on our results is that  $\gamma_\psi(a, 0)$  has clearly only  $\alpha$  dependence at one loop. This was originally observed in [29], where it was claimed that the only  $\alpha$  dependence of  $\gamma_\psi(a, 0)$  was at one loop. Clearly in a nonlinear covariant gauge there is  $\alpha$  dependence beyond one loop which is not unexpected. Finally, in relation to the renormalization of the gauge parameter in other conventions, we record the sum of  $\gamma_A(a)$  and  $\gamma_\alpha(a, z)$  is

$$\begin{aligned}
\gamma_A(a) + \gamma_\alpha(a, z) &= -[2\alpha^2 - 3\alpha + 3]\frac{z}{2\alpha} - [5\alpha - 16]\frac{N_f z a}{3\alpha} \\
&\quad - [21\alpha^3 - 20\alpha^2 - 29\alpha + 60]\frac{z^2}{16\alpha} - [30\alpha^2 - 61\alpha - 31]\frac{N_f z^2 a}{8\alpha}
\end{aligned}$$

$$\begin{aligned}
& - [(140\alpha - 240)N_f^2 + (1215\alpha - 1296\zeta(3)\alpha - 6210 + 5184\zeta(3))N_f] \frac{za^2}{54\alpha} \\
& - [264\zeta(3)\alpha^4 + 370\alpha^4 - 48\zeta(3)\alpha^3 - 116\alpha^3 - 1008\zeta(3)\alpha^2 \\
& \quad - 1093\alpha^2 + 432\zeta(3)\alpha + 1778\alpha + 360\zeta(3) - 367] \frac{z^3}{128\alpha} \\
& + O(z^n a^{4-n})
\end{aligned} \tag{3.6}$$

which is clearly non-zero for  $z \neq 0$ . Moreover, like the MAG, (see, for instance, [15]), the corresponding anomalous dimension is also singular in the  $\alpha \rightarrow 0$  limit, though similarly the remaining renormalization group functions, including the  $\beta$ -functions, are finite in this limit. For  $\gamma_A(a)$ ,  $\gamma_c(a, z)$  and  $\gamma_\psi(a, z)$  this is primarily because in (3.3) and (3.4), the term involving  $\gamma_\alpha(a, z)$  is multiplied by  $\alpha$  and  $Z_A$ ,  $Z_c$  and  $Z_\psi$  themselves are non-singular at  $\alpha = 0$ .

Finally, it might be tempting to try and remove the  $\alpha = 0$  singularity in  $\gamma_\alpha(a, z)$  by a suitable coupling constant redefinition. Whilst this would produce renormalization group functions analytic in  $\alpha$ , one must be careful in ensuring that the original theory is retained. For instance, to remove the  $1/\alpha$  terms in  $\gamma_\alpha(a, z)$  the least one must do is to redefine  $z$  by a factor proportional to  $\alpha$ . In the simplest case, setting  $z = \alpha \bar{z}$  one would formally have non-singular renormalization group functions. However, in this instance returning to (2.3) in the absence of electrons the Landau gauge Lagrangian would then describe self-interacting photons with non-interacting ghosts. This is not consistent with the notion that without electrons the photon is a free field. Therefore given this, avoiding what might be perceived to be a problem in a renormalization group function, which has no physical interpretation, in order to render it analytic, has a significant affect on the nature of the original theory. In other words whilst it might seem unnatural to have a theory with couplings which are singular as  $\alpha \rightarrow 0$  leading to a singular anomalous dimension, the nature of the theory remains consistent.

## 4 Discussion.

We have completed the full three loop renormalization of QED in the nonlinear 't Hooft-Veltman gauge. This extends the one loop calculations of [17, 18]. Whilst the authors of [17, 18] were the first to observe that the longitudinal part of the photon is renormalized unlike in a linear gauge, we have carried out a slightly more general analysis by allowing for a covariant gauge parameter  $\alpha$  and the inclusion of an additional coupling  $\xi$  in order to track the loop calculation in a similar way to the usual coupling constant  $a$ . The coupling  $\xi$  was not initially fixed to a specific value. Consequently the singular renormalization of the gauge parameter emerges. Whilst this is not a new feature of a nonlinear gauge fixing, since the MAG of QCD has the same property, it does not disrupt either the renormalizability of the theory, [16], or the evaluation of gauge independent quantities such as the  $\beta$ -functions. This is primarily because although the gauge in one sense is only defined in the  $\alpha \rightarrow 0$  limit, (2.3), the anomalous dimension of  $\alpha$ , (3.6), has no physical meaning or interpretation.

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